How to Shuffle a Deck

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1 Introduction

Every time you have to shuffle a deck one will tell you "Shuffles it right !". But what does it means ? Are there good and bad ways to shuffle a deck ? Can we do a perfect shuffle ? Those questions will be answered in this article and a method to correctly shuffle a deck of 52 cards will be given.

2 Modelisation

In this section, we propose a mathematical modelisation and a theoretical limit of the number of iteration of the Shannon shuffle.

2.1 Shuffle

Let's model a shuffle as a Random Variable that take a permutation of the symmetric group with n elements as input and output a permutation of the symmetric group with n elements as described in Figure 1 at page 1. Mathematically we define a shuffle S as : $S : \mathfrak{S}_n \to \Omega = \mathfrak{S}_n$. We define a perfect shuffle as a shuffle that have a uniform distribution. Our goal is to find a good approximation of this shuffle that can be done in a short amount of time.

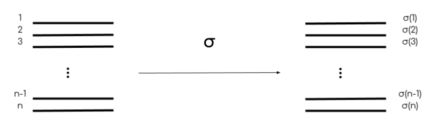

Figure 1: Shuffle diagram

2.2 Shannon shuffle

Shannon shuffle is the scientific modelisation of the American shuffle. Shuffle description :

1. The deck is cut into two smaller decks (deck1 and deck2)

2. One card from deck1 or from deck2 fall creating a new deck (deck3)

3. Step 2 is repeated until deck1 and deck2 are empty

You will find a description of this shuffle in Figure 2 at page 2.

The cut position follows a binomial law K : $\mathfrak{B}(n, \frac{1}{2})$. Let's define $D_{1,i}$: "The card falls from deck 1 at step i" $D_{2,i}$: "The card falls from deck 2 at step i" and $n_{i,j}$ the number of cards of deck j at step i. The probability of those events are : $P(D_{1,i}) = \frac{n_{i,1}}{n-i}$ and $P(D_{2,i}) = \frac{n_{i,2}}{n-i}$

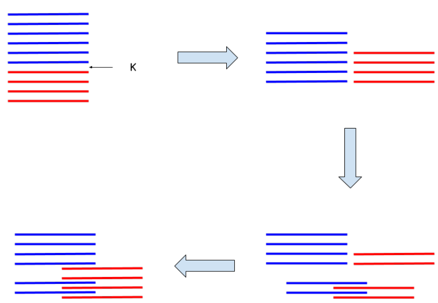

Figure 2: Shannon shuffle diagram

2.3 M-Iterated Shannon shuffle

An m-iterated Shannon shuffle is a composition of m independant Shannon shuffles.

2.4 Number of iterations minimization

This method of iteration minimization is a weaker result than the one described Dr Biane in its article Combien de fois faut-il battre un jeu de cartes?[1].

The probability to have a shuffle is higher than the probability to have a shuffle with a cut position k : $P(S = \sigma) \geq P(K = k) * P_{K=k}(S = \sigma)$. Following step by step the shuffle process, we obtain that $P(K = k) = \binom{n}{k}\frac{1}{2^n}$ and $P_{K=k}(S = \sigma) = \frac{k!(n-k)!}{n!} = \frac{1}{\binom{n}{k}}$. We can conclude that $P(S = \sigma) \geq \frac{1}{2^n}$. After a Shannon shuffle, a maximum of 2^n permutations can be obtained.

After an m-iterated Shannon shuffle, a maximum of 2^{m*n} permutations can be obtained. To have a perfect shuffle the number of permutations that can be obtained must be $n!$. We obtain that $m \geq \frac{log_2(n!)}{n}$, with $n = 52$, $m \geq 5$.

3 Statistic analysis

In this section we will use a python simulation. The number of card of the deck will be $n = 52$. The implemented shuffle is an m-iterated Shannon shuffle. The idea of this section is to use mathematical results about permutations to compare them with the simulation.

3.1 Derangement criterion

Let's find the proportion of derangement in $\mathfrak{S}_n$ ($\frac{D_n}{n!}$). Formal definition of a derangement : A derangement σ of $\mathfrak{S}_n$ is a permutation that fits this property : $\forall i \in [\![1, n]\!], \sigma(i) \neq i$. The ensemble of all the permutations of n elements with k fixed points, $\mathfrak{S}_{n,k}$, is in bijection with $\mathscr{P}_k(E) \times \mathfrak{S}_{n-k}$. We conclude that : $n! = \sum_{k=0}^{n} Card(\mathfrak{S}_{n,k}) = \sum_{k=0}^{n} \binom{n}{k} * D_{n-k}$. With power series and this equality, we find that : $\sum_{n=0}^{\infty} \frac{D_n}{n!}x^n = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(-1)^k}{k!}x^n$ With the unicity of a power serie we obtain :

$$\frac{D_n}{n!} = \sum_{k=0}^{n} \frac{(-1)^k}{k!} \to \frac{1}{e}$$

A numerical simulation gives us the proportion of derangement for m-iterated Shannon shuffle in Figure 3 at page 3. A decrease of the error can be observed with $m > 5$ which is in accordance with the number of iterations minimization but this criterion alone does not help us to determine the quality of a shuffle.

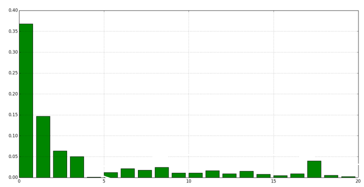

Figure 3: Evolution of the error of the proportion of derangement for m-iterated Shannon shuffle

3.2 Successions criterion

The derangement criterion only gives us a discreet value for each shuffle the succession criterion gives a distribution. A succession i in a permutation σ is an integer that fits this property : $\sigma(i+1) = \sigma(i) + 1$.

To create a permutation with $n+1$ elements and k successions, we can take a permutation of n elements with $k-1$ successions and insert $n+1$ in the only position that add a succession or we can take a permutation of n elements with $k+1$ successions and insert $n+1$ in one of the k positions that break a succession or we can take a permutation of n elements with k successions and insert $n+1$ in one of the $n-k$ positions that keep the number of successions. This reasoning gives us the following formula about the number of permutations of n elements with k successions (S_k^n) :

$$S_k^{n+1} = S_{k-1}^n + k * S_{k+1}^n + (n-k) * S_k^n$$

The succession criterion gives us a better idea of the quality of a shuffle but the distribution is not smooth that why we need a criterion that gives us a smoother distribution. You will find the evolution of the distribution in Figure 4 at page 3.

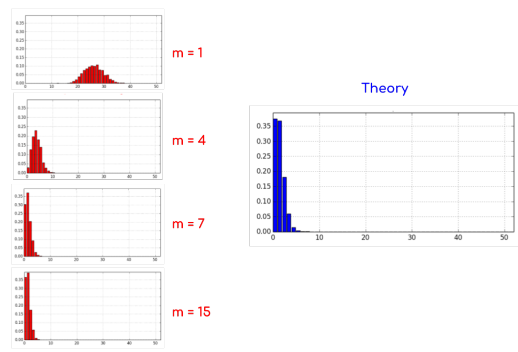

Figure 4: Evolution of the distribution of successions for m-iterated Shannon shuffle

3.3 Rising sequence criterion

We will have the same approach for the rising sequence criterion than for the successions criterion. This criterion is well-known and used by magicians to do their tricks. A rising sequence is a maximal sub-sequence composed of successive number[1]. Every permutation σ can be decomposed uniquely into a juxtaposition of rising sequences. A decrease of σ is an integer i that follows this property $\sigma(i) > \sigma(i+1)$. The juxtaposition of rising sequences is easily found with the decrease of σ^{-1} as you can see in the following example :

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 5 & 1 & 6 & 2 & 9 & 3 & 4 & 8 & 7 \end{pmatrix}$$

$$\sigma^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 4 & 6 & 7 & 1 & 3 & 9 & 8 & 5 \end{pmatrix}$$

The rising sequences of σ are : (1, 2, 3, 4), (5,6,7), (8), (9)
The decrease of σ^{-1} are : 4, 7, 8

The number of rising sequences is equal to the number of decrease of σ^{-1} - 1. To compute the distribution of rising sequences we will compute the distribution of decreases. To create a permutation of $n+1$ elements with k decreases, we can take a permutation of n elements with k decreases and insert $n+1$ in one of the $k+1$ positions that keep the number of decreases or we can take a permutation of n elements with $k-1$ decreases and insert $n+1$ in one of the $n-k$ positions that add one decrease. We obtain the formula to compute the number of decreases D_k^n :

$$D_k^{n+1} = (k+1) * D_k^n + (n-k) * D_{k-1}^n$$

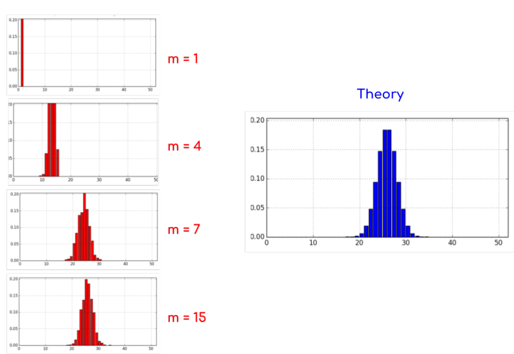

Figure 5: Evolution of the distribution of rising sequences for m-iterated Shannon shuffle

This criterion seems to be good enough to judge the quality of a shuffle because the distribution is smooth and centered. Furthermore, magicians know and use this criterion for their tricks.

3.4 χ^2 analysis

Considering experimental and theoretical distributions of the rising sequences (Exp and Th) as $\mathbb{R}^{53}$ we launch a χ^2 analysis[2] on $N = 1000$ tries on m-iterated Shannon shuffles (m between 1 and 20). The formula applied in this process is :

$$T = \sum_{i=0}^{52} \frac{(N * Exp[i] - N * Th[i])^2}{N * Th[i]}$$

Every m-iterated Shannon shuffles succeed the χ^2 test wit $\alpha = 0.05$ when $m \geq 9$.

4 Conclusion

To conclude a 9-iterated Shannon shuffle is a good shuffle. For a well trained person such shuffle can be realized in less than a minute.

References

[1] Biane, P., 2002. Combien de fois faut-il battre un jeu de cartes?. Gaz. Math, 91(4), p.10.

[2] Pearson, K., 1894. Contributions to the mathematical theory of evolution. Philosophical Transactions of the Royal Society of London. A, 185, pp.71-110.